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Application of Synthesis Techniques to Problems Involving Time Dependence*

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A method described in a previous article for synthesizing three-dimensional flux distributions is here extended to reactor kinetics problems and to lifetime studies. The method is outlined and some numerical examples presented. The results of these show the method to be a practical way of solving time dependent reactor problems with a detailed spatial model.

I. INTRODUCTION

The solution of the few-group neutron diffusion equations in three dimensions becomes very expensive if a detailed mesh is desired. For example, a problem of 50,000 points has been found to require almost a full day of computing time on the Philco S-2000. When time dependence is added, as in a depletion or kinetics study, the situation becomes much worse; time dependent problems are costly even with only one or two space dimensions. Therefore, for core designs requiring solutions to large size problems, there is a pressing need for approximation methods which give reasonably accurate results at greatly reduced cost. A recent paper[1] described several methods of this type which could be used to construct or "synthesize" three dimensional neutron fluxes using only one and two dimensional computer programs. In that paper, results of a rather simple test problem were given. Since then the methods have been applied to full scale problems with considerable success. It is therefore of some interest to extend these methods to depletion and kinetics problems. These extensions are conceptually straight forward and have been considered qualitatively for some time.[2] The important point, and the purpose of this paper, is to report that the methods have been applied to realistic problems, have been found to be practical and to yield results of good accuracy.

In particular, one of the methods has been used to study a three dimensional model of the Shippingport PWR through a large part of its lifetime—including a seed replacement. This paper will describe the results of this test problem in some detail.

II. SPACE SYNTHESIS

We begin by reviewing the idea underlying the time independent or pure "space" synthesis. For this purpose consider a cylindrical reactor divided into two axial zones by the partial insertion of a bank of control rods. It is desired to find the three-dimensional flux distribution using only one- and two-dimensional calculations. In the rods-out zone, far from the rod interface the flux distribution for energy group g assumes an asymptotic shape, $H_1^g(x,y)$, in the horizontal plane. Similarly there is an asymptotic shape, $H_2^g(x,y)$, for the rods-in zone; therefore, it is plausible to seek an approximate solution in the form:

$$\phi^g_{\text{approx}}\ (x,y,z) = Z_1^g(z)H_1^g(x,y) + Z_2^g(z)H_2^g(x,y), \tag{1}$$

the idea being that at any axial level near the zone interface the true x,y shape is well approximated by some mixture of the asymptotic shapes. The proportion of the mixture is given by the values of the $Z_i^g(z)$ at that level. The "trial functions,"

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[1] S. KAPLAN, "Some New Methods of Flux Synthesis," *Nucl. Sci. Eng.* 13, 22-31 (1962).

[2] J. BEWICK, A. F. HENRY, S. KAPLAN, "Synthesis Approximations in the Time Direction", WAPD-T-1463 (April 1962) - also see *Trans. Amer. Nucl. Soc.* 5, 1, p. 177-178, (1962).

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$H_i^g(x,y)$, can be found using two-dimensional programs; thus it remains only to determine the "mixing coefficients" $Z_i^g(z)$. In other words, it remains to select a particular approximation function (hopefully the best one) out of the class of functions of the form (1). This can be done from a variational principle in the manner of Selengut,[3] or alternately by the method of weighted residuals.[4] Our experience has been that the special case of the weighted residual approach known as Galerkin's method yields satisfactory and economical results. In this method the approximate form (1) is substituted into the three dimensional diffusion equations, obtaining for each group a residual function $\epsilon^g(x,y,z)$. These residual functions are then required to be orthogonal to the functions $H_i^g(x,y)$, i.e.,

$$\iint \epsilon^g(x,y,z) H_i^g(x,y)\,dx\,dy = 0 , \qquad (2)$$

and this requirement yields a set of differential equations in z defining the mixing functions. (For details of this process and illustrative examples see References 5 and 6.)

III. TIME SYNTHESIS

The basic idea in the space synthesis is the use of trial shapes which are appropriate to the asymptotic ranges of the Z (axial) variable. In "time synthesis" one uses trial shapes appropriate to asymptotic ranges of the time variable.

To illustrate this application consider a two-dimensional lifetime study. Problems of this type are customarily handled[7] by dividing the reactor into "time steps," assuming the flux shape is constant over each time step, and depleting according to this shape for the period of that time step. This procedure is costly primarily because a new diffusion calculation has to be done at each time step. The synthesis method can be applied to reduce this cost in the following way: first, an ordinary depletion calculation is done, depleting to end-of-life using a small number (e.g., two) of large time steps and saving the flux shape obtained at each step. Then, using these shapes as trial functions, one returns to the beginning-of-life and depletes in small time steps, at each step synthesizing a flux shape in the form of a linear combination of these trial functions.

$$\varphi^g_{\text{approx}}(x,y,t) = T_1^g(t)H_1^g(x,y) + T_2^g(t)H_2^g(x,y) + T_3^g(t)H_3^g(x,y). \qquad (3)$$

A numerical example of this type of synthesis can be found in Reference 6.

IV. SPACE-TIME SYNTHESIS

By applying the space synthesis and time synthesis ideas simultaneously one arrives at a procedure for doing depletion studies of detailed three-dimensional cores. This procedure is described most simply for the case of a reactor operating with all control rods in a bank. For this case, two-dimensional depletion calculations are performed for each axial zone, depleting to end-of-life in one or two large time steps. The x,y flux shapes obtained from these calculations at each time step are then all used together as trial functions in a spatial synthesis of the three-dimensional core at the beginning-of-life. With the flux so obtained the reactor is depleted three-dimensionally for one small time step. At the end of this time step a new spatial synthesis is performed with the same set of trial functions and the process repeated.

For this simultaneous space and time synthesis the approximate form of solution is:

$$\varphi^g_{\text{approx}}(x,y,z,t) = Z_1^g(z,t)H_1^g(x,y) + \ldots + Z_n^g(z,t)H_n^g(x,y). \qquad (4)$$

V. NUMERICAL EXAMPLE OF SPACE-TIME SYNTHESIS

The nuclear design group responsible for the Shippingport pressurized water reactor has performed a three-dimensional depletion calculation for their reactor, using the model shown in Figure 1, and the TNT-1 program.[7] As a test of the method this calculation was repeated using the synthesis approximation. In this problem the rods are withdrawn one group at a time. Table I shows the time steps used for this calculation, and indicates the position of the controlling rod group at each time step.

[3] D. S. SELENGUT, "Variational Analysis of Multidimensional Systems," HW59126, *Quarterly Physics Report* (October-December 1958).

[4] S. H. CRANDALL, *Engineering Analysis*, McGraw-Hill New York (1956).

[5] W. R. CADWELL, S. KAPLAN, O. J. MARLOWE, "Equations and Programs for Synthesis Approximations," WAPD-TM-377 (April 1963).

[6] S. KAPLAN, J. BEWICK, "Space and Time Synthesis by the Variational Method," WAPD-BT-28 (Bettis Technical Review).

[7] O. J. MARLOWE, "Nuclear Reactor Depletion Programs for the Philco-2000 Computer," WAPD-TM-221.

TABLE I

PWR Test Problem

(power level of reactor is 231 megawatts)

Time Step	No. of Full Power Hours at Beginning of Time Step	Fraction of U^{235} Remaining in Seed at end of Time Step	Controlling Rod Group and Position (inches above bottom of core)	Eigenvalue ,λ, by TNT-1	Eigenvalue ,λ, by Synthesis	Fraction of Power in Blanket by TNT-1	Fraction of Power in Blanket by Synthesis
Seed 1							
0	0	0.99475	I - 54	.9738	.9732	.4864	.4863
1	62.3	0.98870	II - 21	.9687	.9688	.4765	.4740
2	133	0.92591	II - 24	.9721	.9720	.4780	.4759
3	897	0.85754	II - 33	.9735	.9729	.5133	.5115
4	1800	0.77691	II - 42	.9763	.9756	.5248	.5248
5	2905	0.70173	II - 69	.9840	.9828	.5271	.5275
6	3976	0.62776	III - 51	.9742	.9726	.5731	.5731
7	5076	0.58079	IV - 45	.9718	.9701	.5857	.5872
8	5805	0.57406	IV - 72	.9698	.9680	.5821	.5844
Seed 2							
9	5805	0.99671	I - 30	.9607	.9596	.5196	.5195
10	5856	0.98762	I - 69	.9718	.9706	.5055	.5055
11	5990	0.94040	II - 9	.9673	.9653	.5076	.5076
12	6705	0.85989	II - 15	.9659	.9636	.5133	.5135
13	7970	0.79326	II - 24	.9645	.9616	.5260	.5270
14	9050	0.72960	II - 36	.9607	.9581	.5422	.5435
15	10,125	0.67629	II - 54	.9623	.9599	.5500	.5503
16	11,035	0.62144	III - 30	.9610	.9597	.5546	.5537
17	11,983	0.58644	III - 63	.9615	.9599	.5691	.5686
18	12,603	0.54565	IV - 39	.9615	.9603	.5714	.5715

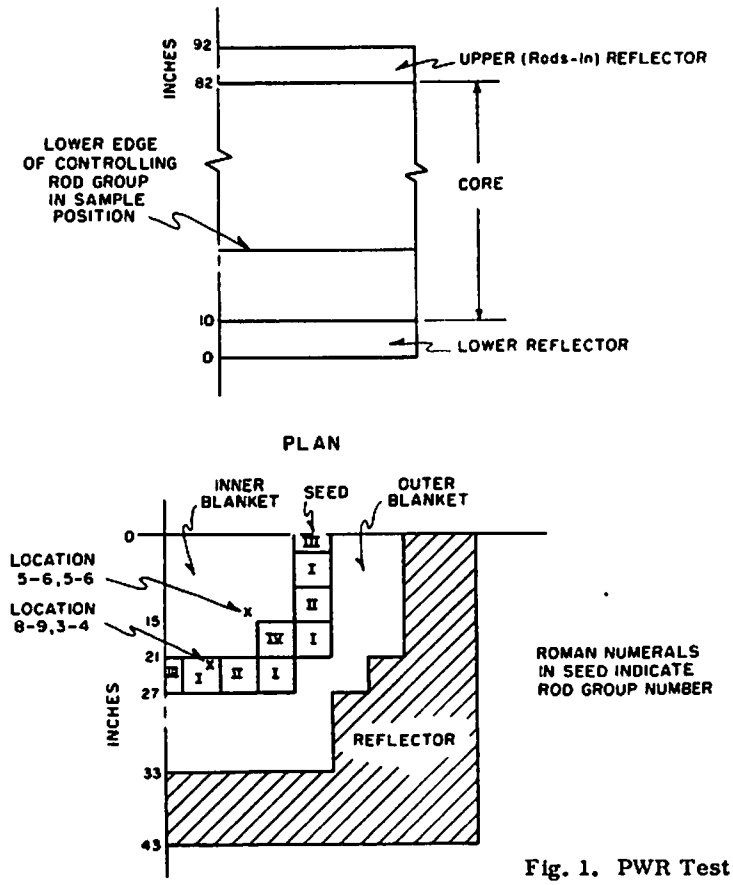


Fig. 1. PWR Test Problem Geometry.

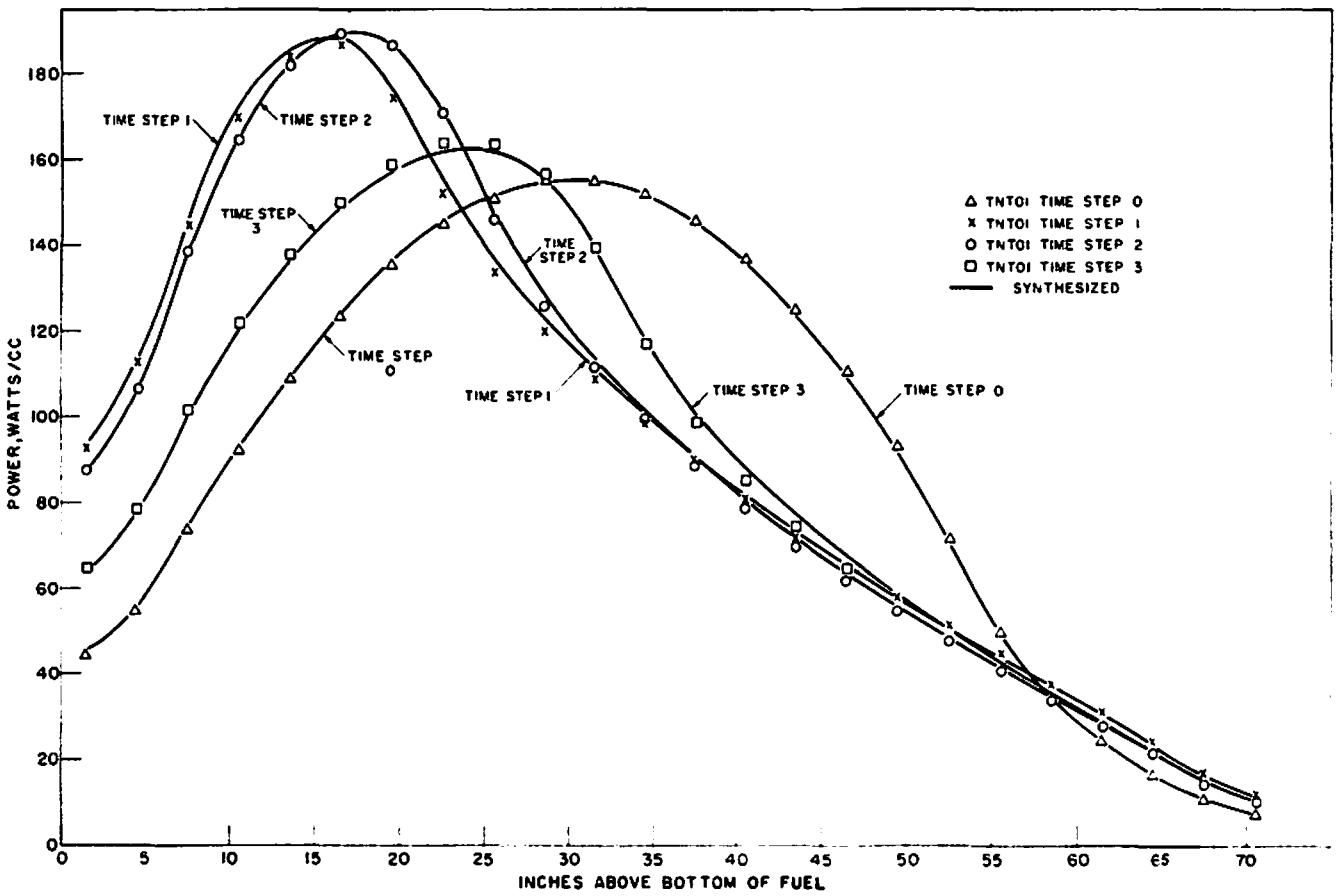


Fig. 2. Axial Power Distribution at Location 8–9, 3–4.

Because of the programmed rod operation the strategy involved in obtaining trial functions for the synthesis is considerably more complicated than that outlined in the previous section and will not be described here. The basic motive, however, was to have in the "arsenal" of trial functions at least two shapes corresponding to each control rod configuration; one of these representing a lightly depleted plane and the other a highly depleted plane.

Table I shows a comparison of the eigenvalue or multiplication constant as obtained by the synthesis and the exact calculation. It also compares the power sharing between seed and blanket as predicted by the two methods. The accuracy of the power distribution calculated by the synthesis is indicated in Figures 2-9. These figures show the axial power distributions as a function of time at two representative x,y positions. In plotting these curves the synthesized and exact solutions are normalized so that the total power of the three-dimensional core is the same by each method. The different x,y locations are not normalized individually.

This model of the PWR is relatively crude. In particular, the control rods are not represented explicitly in the mesh but instead are represented by a uniform poison in the seed subassemblies. This model was chosen to test the synthesis because an exact calculation was available. With the synthesis it would be very feasible to use a much more detailed model including explicit rods, whereas the cost of doing this with a direct calculation would be prohibitive.

The solution of the present test problem by the exact method required approximately 1 hour of computer time on the Philco S-2000 for each time step in the problem. The synthesis required about one-third as much. However, the time required for the synthesis increases roughly linearly with the number of mesh points whereas the time for an exact solution increases roughly as the square of the number of points. Therefore the time advantage of the synthesis is much greater for larger problems. Further, with the synthesis it is possible to search for critical rod positions at essentially no additional cost whereas this is very costly by the exact method. It is also possible, at

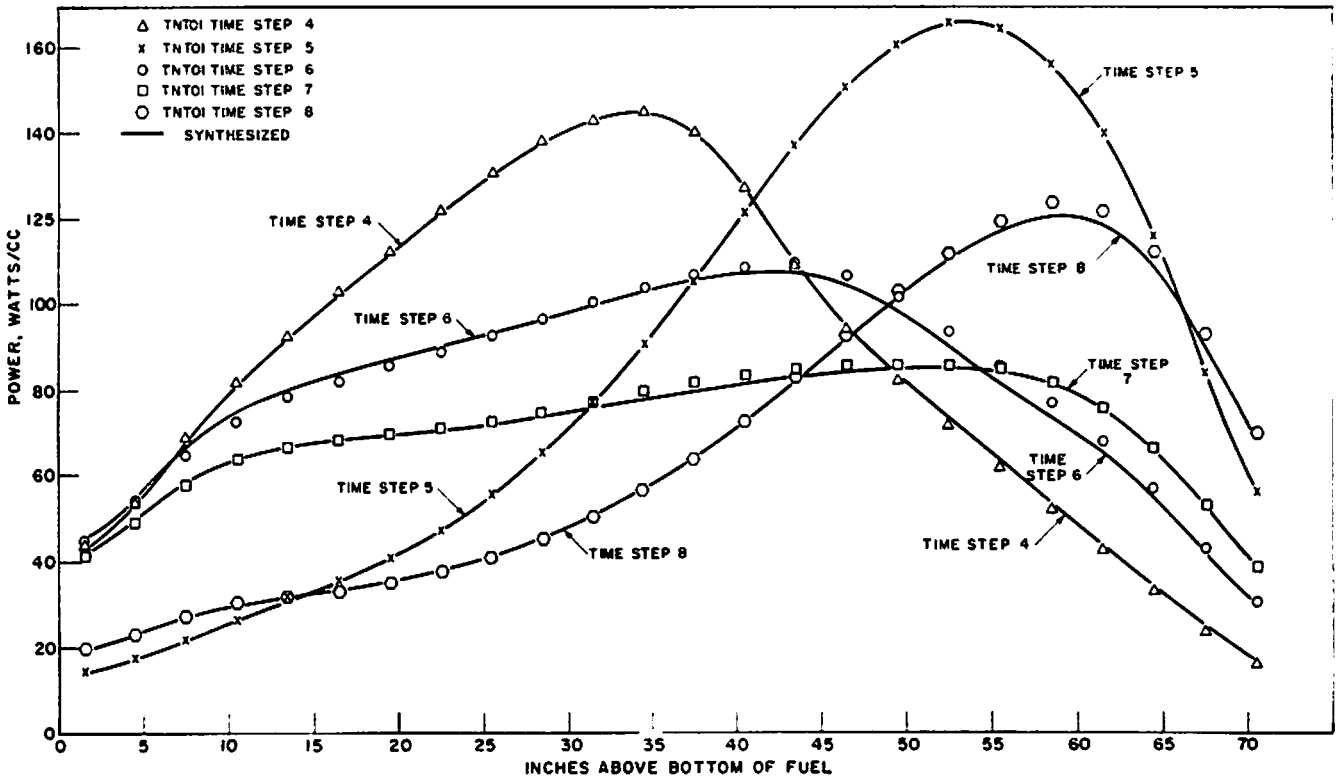


Fig. 3. Axial Power Distribution at Location 8-9, 3-4.

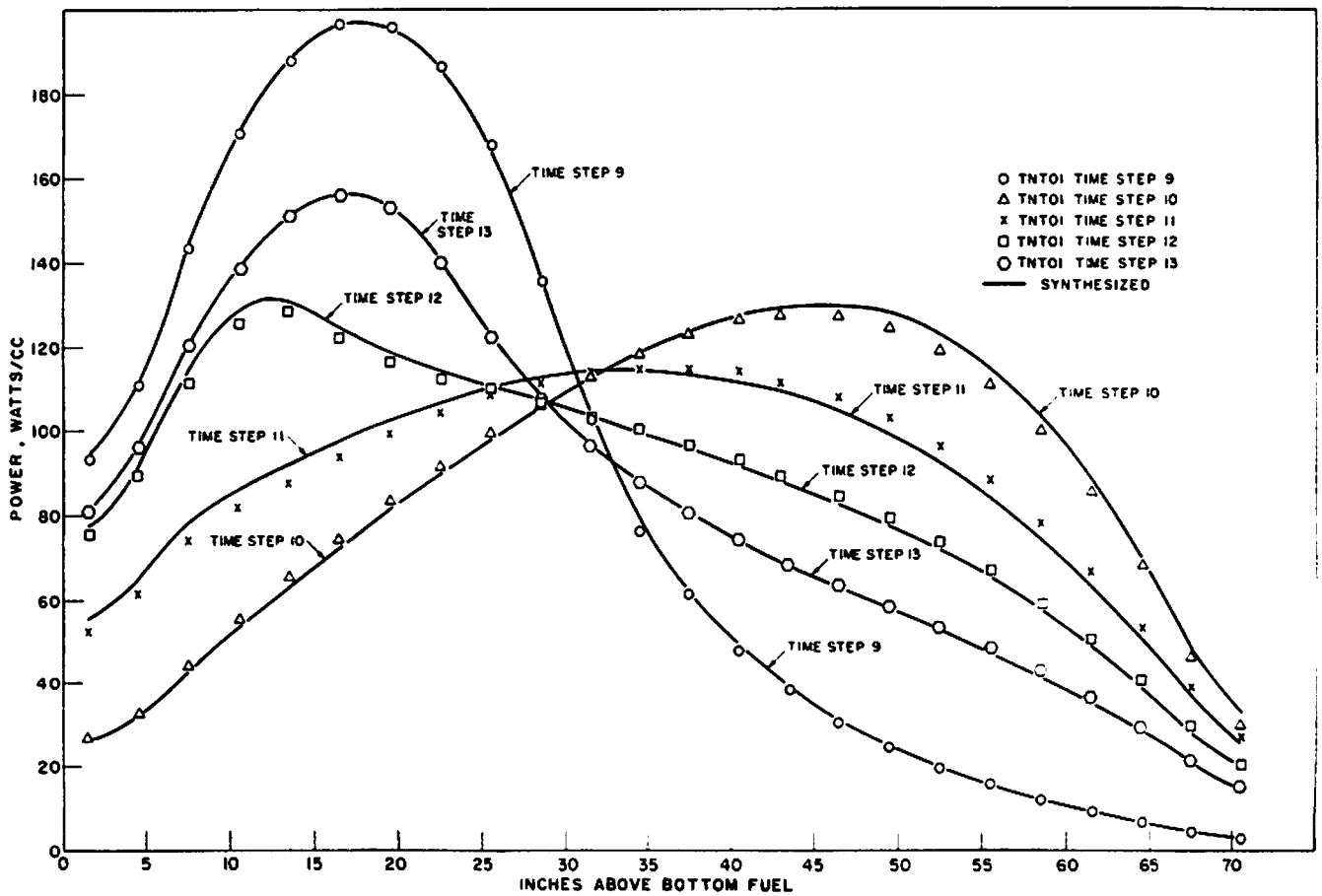


Fig. 4. Axial Power Distribution at Location 8-9, 3-4.

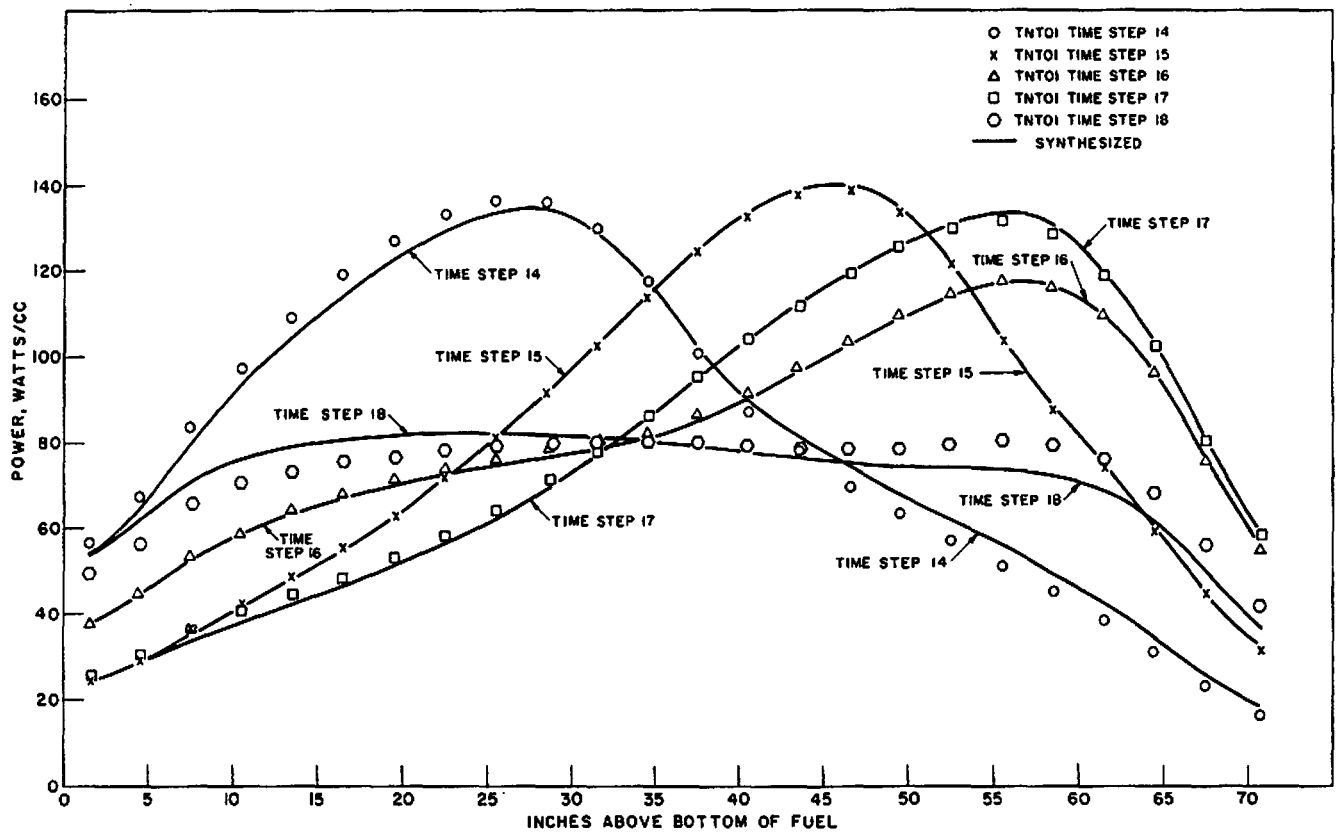


Fig. 5. Axial Power Distribution at Location 8-9, 3-4.

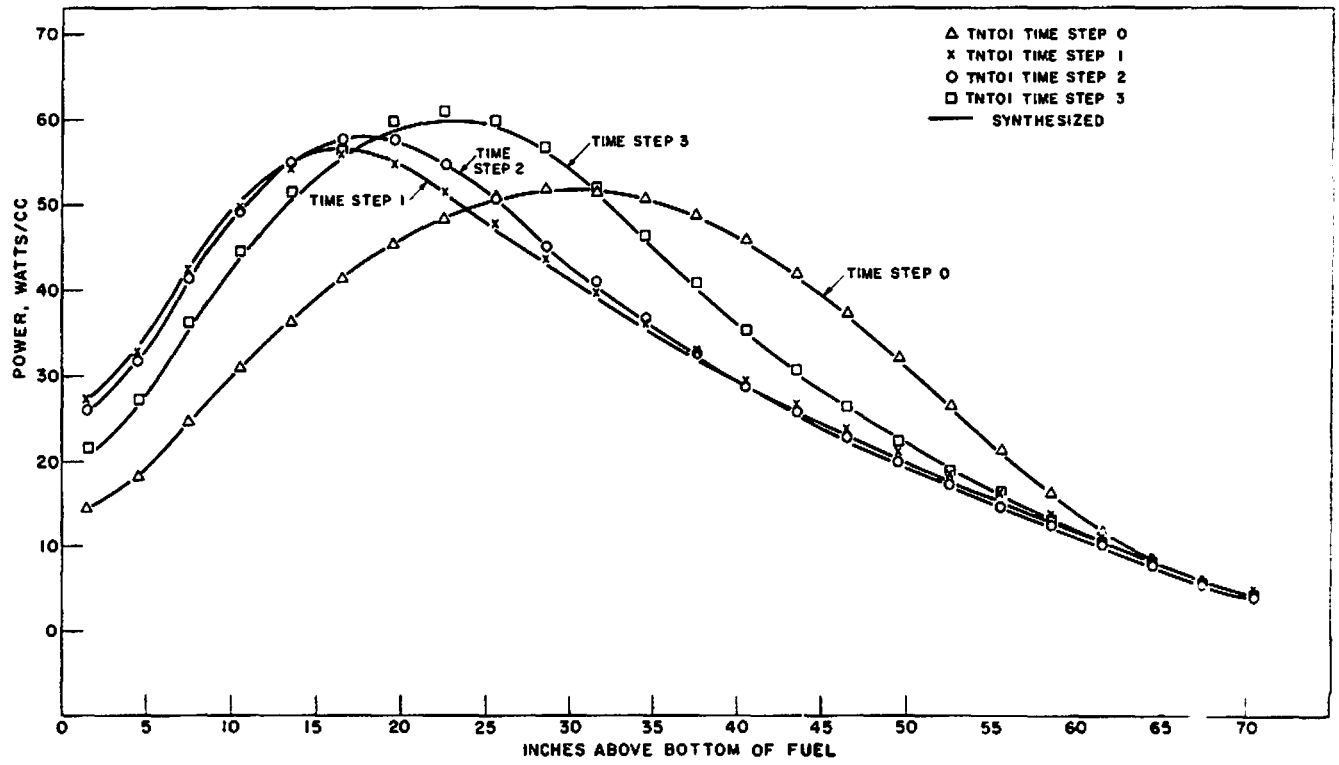


Fig. 6. Axial Power Distribution at Location 5-6, 5-6.

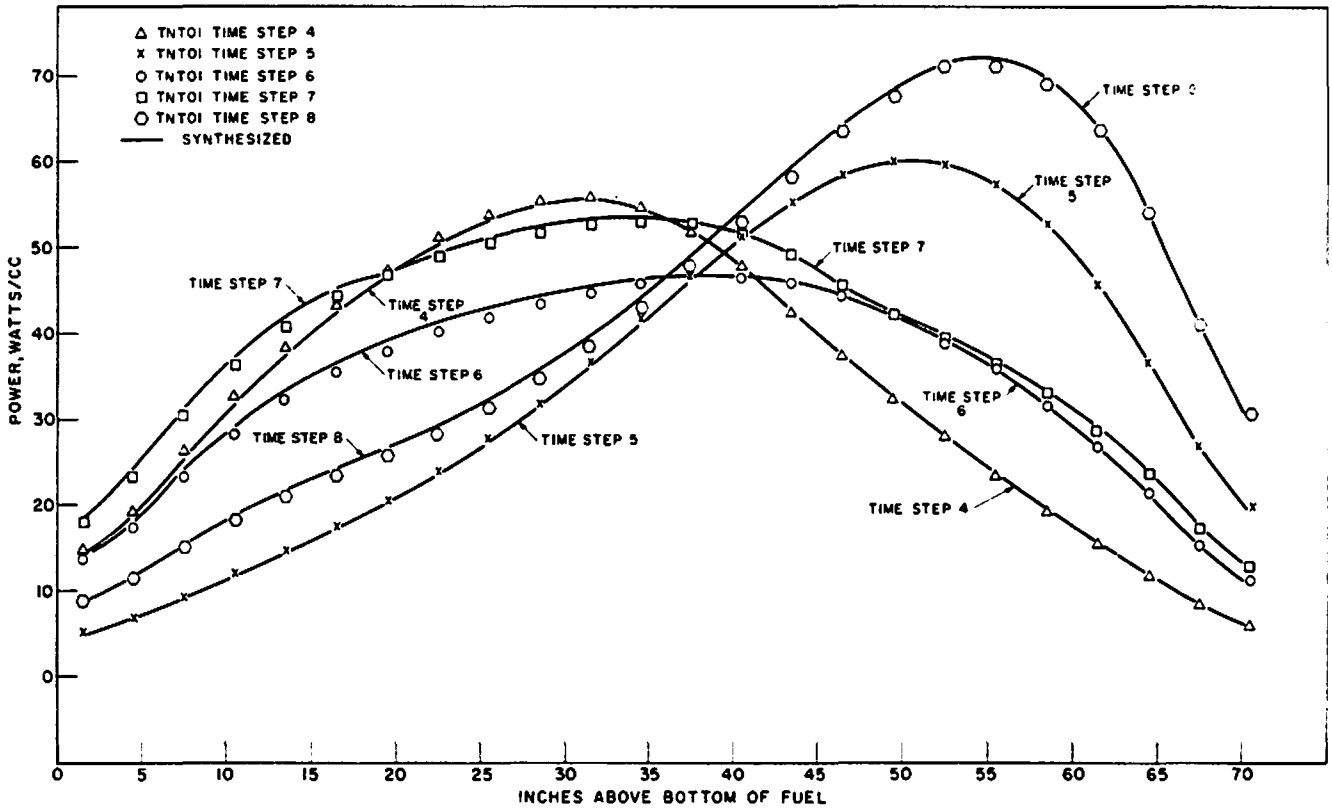


Fig. 7. Axial Power Distribution at Location 5-6, 5-6.

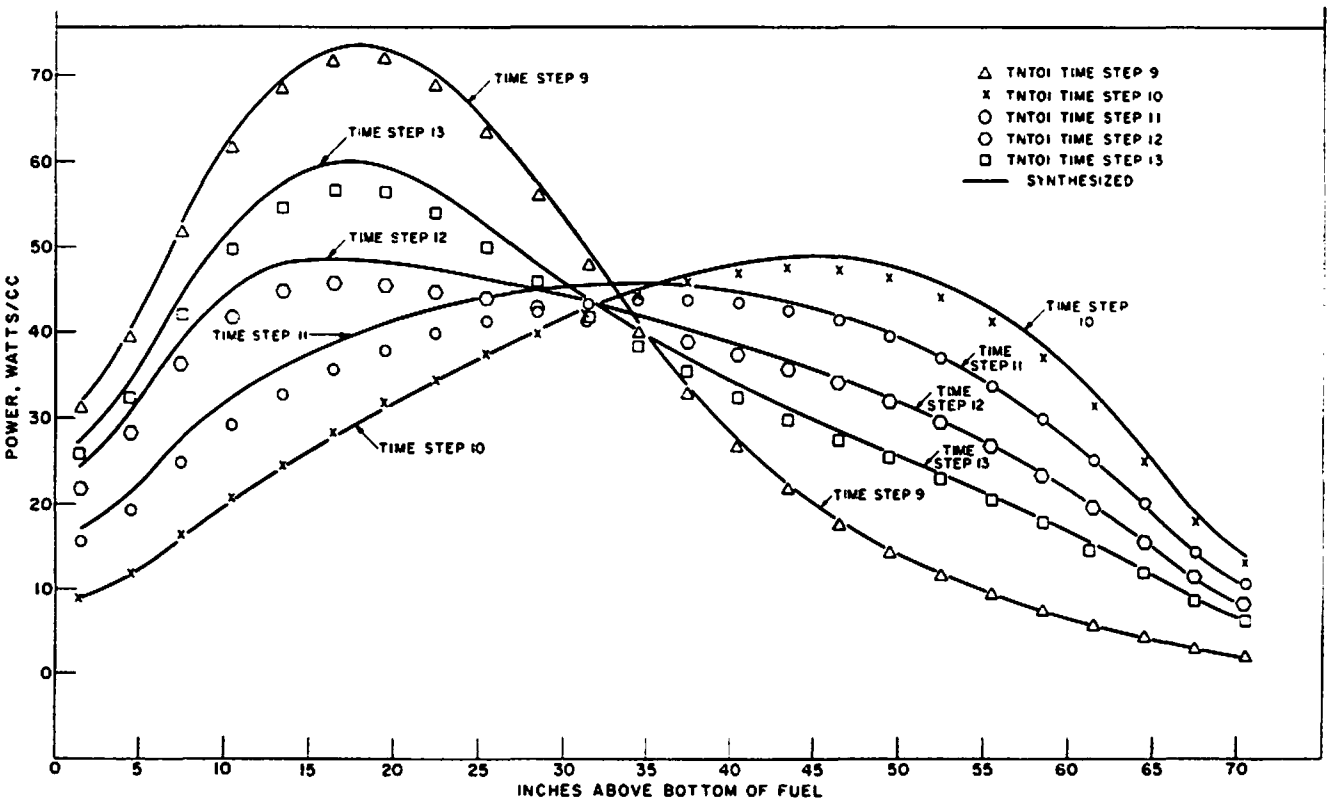


Fig. 8. Axial Power Distribution at Location 5-6, 5-6.

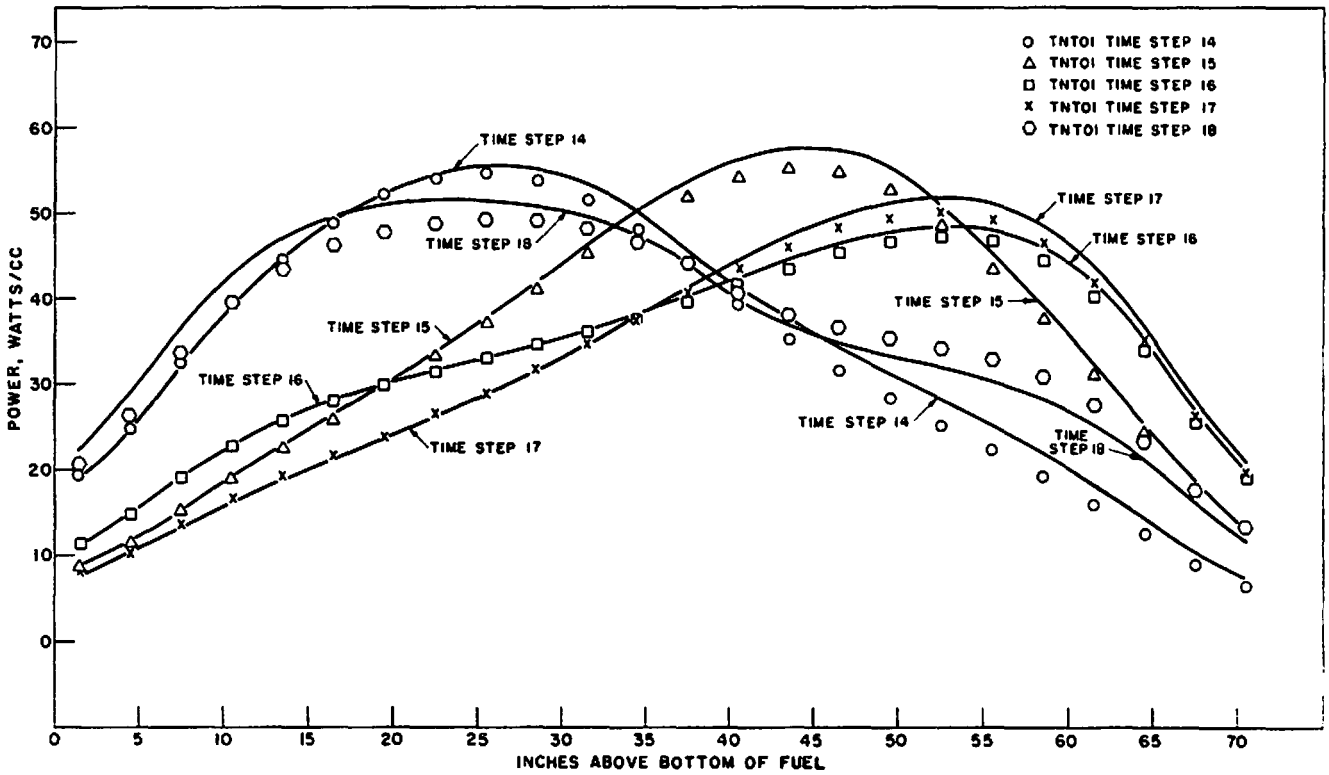


Fig. 9. Axial Power Distribution at Location 5-6, 5-6.

little extra cost, to synthesize a three-dimensional adjoint flux for use in perturbation calculations.

With the synthesis method reducing the cost of obtaining fluxes, the main obstacle to performing depletion studies on detailed, three-dimensional cores, is the time required to calculate the change in each isotopic density at each mesh point. To reduce this time, the tactic that immediately comes to mind is to treat blocks of mesh points as single points for depletion purposes: in other words, to deplete all points in the block by the same flux. If a particular point is of special interest, a return may be made to beginning of life and that point depleted specifically using the flux history at that point obtained from the block-depletion of the whole core.[a]

This block approximation not only reduces the time required to do the depletion portion of the calculation but also substantially reduces the time required to do the spatial synthesis at each time step. Thus a block depletion enhances the time advantage of the synthesis over the exact method.

VI. OTHER TYPES OF TIME SYNTHESIS

Thus far we have discussed time synthesis in connection with depletion problems. Such problems involve a long time scale so that one ignores the time derivatives in the diffusion equations or, equivalently, uses a time independent variational principle as in Equation (1).

To illustrate application of the synthesis idea to kinetics problems consider the flux tilt transient resulting from a sudden insertion of reactivity on one side of a reactor. Intuitively it may be expected that at any time during the transient the actual spatial shape of the flux can be well approximated by some linear combination of the initial unperturbed shape and the final asymptotic shape. Thus the synthesized solution takes the form:

$$\phi^g_{\text{approx}}\ (\underline{r},t) = T^g_1(t)H^g_1(\underline{r}) + T^g_2(t)H^g_2(r) \qquad (5)$$

$$C_{\text{approx}}\ (\underline{r},t) = \tau_1(t)\xi_1(\underline{r}) + \tau_2(t)\xi_2(r) \qquad (6)$$

where $H^g_1(\underline{r})$, $H^g_2(r)$ are the initial and asymptotic flux shapes, $C(\underline{r},t)$ is the concentration of delayed neutron precursors. The $\xi_i(r)$ are trial functions appropriate to the precursor distribution. For example, a good choice is

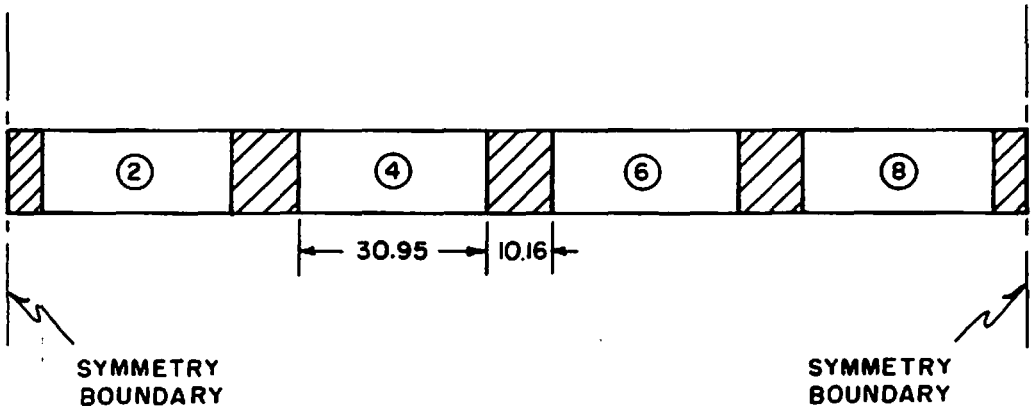
$$\xi_i(r) = \sum_g \Sigma^g_f(\underline{r})\, H^g_i(\underline{r}) \qquad (7)$$

where $\Sigma^g_f(\underline{r})$ is the fission cross section for group g.

[a]This idea was suggested and developed by R. J. Breen at Bettis Laboratory.

To find the mixing coefficients, $T_i^g(t)$ and $\tau_i(t)$, in the synthesized solution one may apply the weighted residual process to the time dependent diffusion equations or, alternatively, use a time dependent variational principle. We include some of the details of the variational approach in the appendix.

The feature that really distinguishes this time synthesis from more conventional modal analyses (e.g., References 10 and 11) however, is not the variational or weighted residual formulation but the use of the initial and asymptotic space shapes as expansion functions as opposed to the use of a set of orthogonal or biorthogonal eigenfunctions. To evaluate this feature a series of test problems have been run on the following one-dimensional model:



"Rodded Ring" Reactor Model.

which is in effect an array of cells arranged into a ring and represents an annular reactor with little coupling across the ring.

Each cell in the model contains a "rod-in" region (crosshatched) and a "rod-out" region. At time zero the reactor is perturbed by decreasing the thermal absorption in region (2) and increasing it by the same amount in region (8). The resulting flux tilt transient was calculated "exactly" using a space-time program and then approximately by the synthesis method and by two modal analysis methods. The first modal analysis used the first two "lambda" modes, that is the first two eigenfunctions of[b]

$$[-\nabla \cdot D\nabla + A]\psi_n = \frac{1}{\lambda_n} \chi F^T \psi_n \,. \tag{8}$$

The second modal analysis used "omega" modes, which are the eigenfunctions of

$$[-\nabla \cdot D\nabla + A - \chi F^T]\psi_n = \omega_n V^{-1} \psi_n \tag{9}$$

where in Equations (8) and (9) the operators are those of the unperturbed reactor. The notation is defined in the appendix.

For small perturbations all three approximate methods give very good results. For large perturbations the synthesis method is superior as indicated by the results shown in Figure 10 and Table II. Figure 10 shows the flux as a function of time after the perturbation and Table II lists the asymptotic periods. For this example two energy groups and one group of delayed neutrons were used.

In general, the synthesis approach has two major advantages over the modal analysis method. First, the trial functions in the synthesis approach can usually be found easily whereas it is often difficult to calculate the appropriate higher eigenfunctions for a modal analysis. Second, the trial functions may be specifically tailored to the problem at hand so that only a few trial functions are needed whereas many eigenfunctions may be required for an adequate approximation.

The examples presented in this section are completely linear problems; that is they contain no feedback from the perturbed flux to the nuclear constants. An example involving xenon feedback is given in Reference 6; however, important problems

[8]D. E. DOUGHERTY and C. N. SHEN, "The Space-Time Neutron Kinetic Equations Obtained by the Semi-Direct Variational Method," *Nucl. Sci. Eng.* 13, 141-148 (1962).

[9]J. LEWINS, "Variational Representations in Reactor Physics Derived from a Physical Principle," *Nucl. Sci. Eng.* 8, 95-104 (1960).

[10]A. FODERARO, H. L. GARABEDIAN, "Two-Group Reactor Kinetics," *Nucl. Sci. Eng.* 14, 22-29 (1962).

[11]K. MOCHIZUKI, A. TAKEDA, "An Analysis of Neutron Flux Spatial Oscillations Due to Xenon Buildup in a Large Power Reactor Core," *Nucl. Sci. Eng.* 7, 336-344 (1960).

[b]The notation used in Equations (8) and (9) is defined in the appendix.

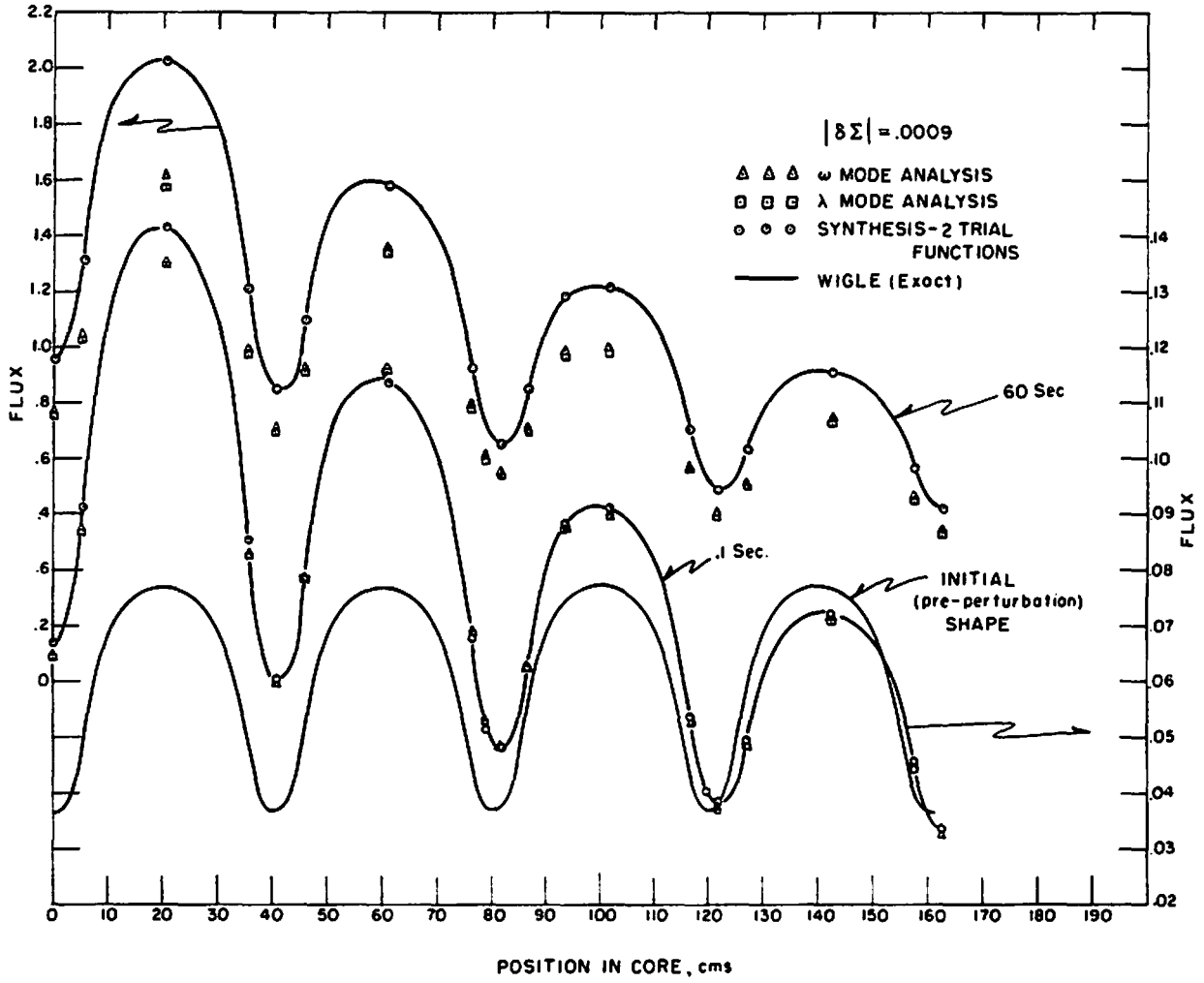


Fig. 10. Flux Tilt Test Problem in Rodded Ring Reactor.

in reactor space-time kinetics involve feedback through changes in the temperature and density of the reactor materials. Severe space-time transients might be encountered, for example, in rocket reactors with hydrogen propellent. For problems of this type a synthesis treatment of the neutron equations, incorporated into a program or analog simulation describing the heat transfer and coolant flow, should be most useful.

TABLE II

Asymptotic Periods for Flux Tilt Test Problem

Method	Period (seconds)
Exact	22.78
"ω" modes	24.81
"λ" modes	24.96
Synthesis	22.82

VII. SUMMARY

In this paper and in Reference 1 we have outlined a class of "synthesis" approximation methods which are being developed to help the nuclear designer to perform more effectively and economically his major task, i.e., the prediction of the reactivity and power distribution at all times during the reactor life. The examples we have presented, and others, demonstrate that the methods can be successfully carried out in practice, that they yield approximate solutions of good accuracy, and that they are flexible enough to handle such items as partial refueling of the core midway through reactor life. Thus there are grounds for hoping that the methods will be of considerable practical value. However, it is important also to point out the pitfalls and weaknesses in the methods.

First of all these synthesis methods fall into the category of "Rayliegh-Ritz type" methods and

share a common characteristic of methods of this type; namely, that in order to obtain good results one must make a good choice of trial functions. In the reactor applications this often requires considerable judgment and intuition; however, it is an art which is quickly learned by experience with a given reactor configuration.

Secondly, serious numerical difficulties can arise if the set of trial functions is close to being linearly dependent. However, these difficulties may be overcome or avoided by various well known stratagems such as orthogonalization. A more quantitative discussion of this subject may be found in Reference 5.

Perhaps the most serious deficiency in the synthesis method is the lack of meaningful error bounds. In this respect the classical Rayliegh-Ritz methods applied, say, to finding the natural frequencies of a vibrating body are superior. However, the variational principles of reactor theory, being associated with non-self adjoint operations, are not maximum or minimum principles and do not yield such bounds. It is even risky to apply the time honored tactic of adding another term to the trial form and seeing how much difference this makes. In the reactor case there is no guarantee that adding terms will not make the answer worse.

In this circumstance all that can be done to gain confidence in the synthesis method, is to compare with exact solutions and with experiment as much as possible. The comparisons obtained thus far, typified by those shown in the present paper, are very encouraging.

APPENDIX: A VARIATIONAL PRINCIPLE FOR TIME DEPENDENT GROUP DIFFUSION THEORY

This appendix describes a variational principle which corresponds to the neutron group diffusion equations and to the adjoint group diffusion equations. The treatment is entirely formal and closely follows that of Morse and Feshbach.[13] However it differs somewhat from the treatments given by Dougherty and Shen[8] and Lewins.[9]

We first define the notation. Let G denote the number of energy groups in the model, and let $S^{j,g}$ denote the fraction of delayed neutrons of type j which are born into energy group g. Then the column matrix

$$S^j = \begin{bmatrix} S^{j,1} \\ S^{j,2} \\ \vdots \\ S^{j,G} \end{bmatrix} \tag{A-1}$$

represents the emergence spectrum for delayed neutrons of type j. Similarly, let

$$X = \begin{bmatrix} X^1 \\ \vdots \\ X^G \end{bmatrix} \tag{A-2}$$

represent the emergence spectrum for prompt neutrons, and let

$$F = \begin{bmatrix} \nu\Sigma_f^1 \\ \vdots \\ \nu\Sigma_f^G \end{bmatrix} \tag{A-3}$$

be a matrix of ν times the fission cross sections. Let D be a diagonal matrix of diffusion constants, and let A be the matrix of absorption and transfer cross sections. Let Φ be the column matrix of group fluxes and C^j the (scalar) concentration of delayed neutron precursors of type j. Finally, let

$$V^{-1} = \text{diag}\left(\frac{1}{V^1}, \frac{1}{V^2}, \ldots \frac{1}{V^G}\right), \tag{A-4}$$

where V^g is the average velocity of neutrons in group g.

By analogy with Hamilton's principle, a variational principle for the time dependent reactor may be stated as follows:

> Between the two instants of time, t_a and t_b, the reactor system (that is, the flux, the precursors and the adjoints) will behave (move) in such a way as to make the integral in Equation (A-5) below, stationary with respect to arbitrary neighboring motions of the system which pass through the same end points (at t_a and t_b) as the actual motion.

[12] J. LEWINS, "The Time-Dependent Importance of Neutrons and Precursors," *Nucl. Sci. Eng.* 7, 268-274 (1960).

[13] MORSE and H. FESHBACH, "Methods of Theoretical Physics," Vol. I, p. 298, 313.

[14] R. COURANT and D. HILBERT, "Methods of Mathematical Physics," Interscience Publishers, Vol. I, p. 249-250.

$$0 = \delta \int_{t_a}^{t_b} \int_R \left\{ - \nabla \Phi^{*T} \cdot D \nabla \Phi - \Phi^{*T} \left[A - (1-\beta) \chi F^T \right] \Phi - \frac{1}{2} \left[\Phi^{*T} V^{-1} \frac{\partial}{\partial t} \Phi - \Phi^T V^{-1} \frac{\partial}{\partial t} \Phi^* \right] \right.$$

$$\left. + \sum_{j=1}^{6} \left[\Phi^{*T} \lambda^j S^j C^j + C^{j*} \beta^j F^T \Phi - C^{j*} \lambda^j C^j - \frac{1}{2} \left(C^{j*} \frac{\partial}{\partial t} C^j - C^j \frac{\partial}{\partial t} C^{j*} \right) \right] \right\} dR dT \tag{A-5}$$

The Euler equations of the principle are the time dependent group diffusion equations

$$[\nabla \cdot D \nabla - A] \Phi(\underline{r},t) + (1-\beta) \chi F^T \Phi(\underline{r},t) + \sum_{j=1}^{6} \lambda^j S^j C^j(\underline{r},t) = V^{-1} \frac{\partial}{\partial t} \Phi(\underline{r},t),$$

$$\beta^j F^T \Phi(\underline{r},t) - \lambda^j C^j(\underline{r},t) = \frac{\partial}{\partial t} C^j , \tag{A-6}$$

and the time dependent adjoint equations

$$[\nabla \cdot D \nabla - A^T] \Phi^*(\underline{r},t) + (1-\beta) F \chi^T \Phi^*(\underline{r},t) + \sum_{j=1}^{6} \beta^j F C^{j*}(\underline{r},t) = -V^{-1} \frac{\partial}{\partial t} \Phi^*(\underline{r},t),$$

$$\lambda^j S^{jT} \Phi^*(\underline{r},t) - \lambda^j C^{j*}(\underline{r},t) = - \frac{\partial}{\partial t} C^{*j} , \tag{A-7}$$

where the superscript T denotes a transpose. The physical interpretation of the time dependent adjoint flux is discussed in several very interesting papers by Lewins.[9,12]

It is now desirable to approximate the true flux and precursor functions as a linear combination of known space shapes and to use this variational principle to determine the time dependent coefficients of combination; in other words, to use the principle to pick the "best" approximating function out of a certain subspace of approximating functions. At this point, however, a conceptual difficulty[c] arises.[14] Since the integral is stationary with respect to neighboring functions having the same end conditions (i.e., at t_a and t_b) as the true solution, it appears that only approximating subspaces which consist of functions having these same end conditions may be used. However, in general, the conditions for the true solution at both t_a and t_b are not known; only the initial conditions are known.

As a possible way around this difficulty we advance the argument that the requirement of having the same end conditions is not really an admissibility condition for the variational problem. The admissible space is a larger space out of which the principle selects that function which makes the integral stationary with respect to variations having the same endpoints as the selected function. In the same way, given an arbitrary approximating subspace, the principle may be used to select that function in the subspace which makes the integral stationary with respect to neighboring functions in the subspace having the same endpoints as the selected function.

Assuming, then, that the principle may be used in this way, approximate solutions are chosen in the form

$$\Phi(\underline{r},t) \approx \sum_{k=1}^{K} H_k(\underline{r}) T_k(t) ;$$

$$C^j(\underline{r},t) \approx \sum_{k=1}^{K^j} \xi_k^j(\underline{r}) \tau_k^j(t), \tag{A-8}$$

$$\Phi^*(\underline{r},t) \approx \sum_{k=1}^{K} H_k^*(\underline{r}) T_k^*(t);$$

$$C^{j*}(\underline{r},t) \approx \sum_{k=1}^{K^j} \xi_k^{j*}(\underline{r}) \tau_k^{j*}(t), \tag{A-9}$$

where the H, H^*, ξ, ξ^* are known trial functions and the T's and τ's are to be determined. Putting these forms into the functional and making the result stationary with respect to variations in T_k^* and τ_k^{j*} gives the following set of ordinary differential equations:

$$\sum_{k=1}^{K} \left\{ \left[-(DB^2)_{ik} - A_{ik} + (1-\beta) M_{ik} \right] T_k \right\}$$

$$+ \sum_{j=1}^{6} \left\{ \lambda^j \sum_{k=1}^{K^j} S_{ik}^j \tau_k^j \right\} = \sum_{k=1}^{K} V_{ik}^{-1} \frac{d}{dt} T_k \tag{A-10}$$

[c] This is an addition to the conceptual difficulty which is also present in the time independent case,[1] and which stems from the fact the variational principle is not a maximum or minimum principle.

$$\beta^j \sum_{k=1}^{K} F_{ik}^{jT} T_k - \lambda^j \sum_{k=1}^{K^j} I_{ik}^j \tau_k^j = \sum_{k=1}^{K^j} I_{ik}^j \frac{d}{dt} \tau_k^j \qquad \text{(A-11)}$$

where

$$(DB^2)_{ik} = \int_R \nabla H_i^{*T} \cdot D \nabla H_k dR \qquad \text{(A-12)}$$

$$A_{ik} = \int_R H_i^{*T} A H_k dR \qquad \text{(A-13)}$$

$$M_{ik} = \int_R H_i^{*T} X F^T H_k dR \qquad \text{(A-14)}$$

$$S_{ik}^j = \int_R H_i^{*T} S^j \xi_k^j dR \qquad \text{(A-15)}$$

$$V_{ik}^{-1} = \int_R H_i^{*T} V^{-1} H_k dR \qquad \text{(A-16)}$$

$$F_{ik}^{jT} = \int_R \xi_i^{j*} F^T H_k dR \qquad \text{(A-17)}$$

$$I_{ik}^j = \int_R \xi_i^{j*} \xi_k^j dR \qquad \text{(A-18)}$$

Equations (A-10) and (A-11), together with initial conditions, serve to define the desired coefficients of combination.

The notation in Equations (A-8) and (A-9) is intended to cover several situations. For example, suppose that there are four energy groups and it is desired to approximate each group flux as a separate linear combination. H_k is then taken to be a diagonal matrix and T_k a column as follows:

$$\Phi(\underline{r},t) = \begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{bmatrix} \approx \sum_{k=1}^{K} \begin{bmatrix} H_k^1 & & & \\ & H_k^2 & & \\ & & H_k^3 & \\ & & & H_k^4 \end{bmatrix} \begin{bmatrix} T_k^1 \\ T_k^2 \\ T_k^3 \\ T_k^4 \end{bmatrix} \qquad \text{(A-19)}$$

On the other hand, suppose that the use of the same coefficients of combination in each group is deemed to be adequate, then the H_k are columns and the T_k are scalars:

$$\begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{bmatrix} \approx \sum_{k=1}^{K} \begin{bmatrix} H_k^1 \\ H_k^2 \\ H_k^3 \\ H_k^4 \end{bmatrix} T_K \qquad \text{(A-20)}$$

Alternately, the choice may be to tie only the top two groups together; then the form is

$$\begin{bmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{bmatrix} \approx \sum_{k=1}^{K} \begin{bmatrix} H_k^1 & 0 & 0 \\ H_k^2 & 0 & 0 \\ 0 & H_k^3 & 0 \\ 0 & 0 & H_k^4 \end{bmatrix} \begin{bmatrix} T_k^1 \\ T_k^3 \\ T_k^4 \end{bmatrix} \qquad \text{(A-21)}$$

and so on. In each case the form chosen for Φ^* should be the same as that for Φ.

If the analysis is restricted to the case of Equation (A-20), then Equations (A-10) and (A-11) simplify, in that all the unknowns and all the coefficients, Equations (A-12) to (A-18), become scalars. Suppose, in addition, precursor trial functions are chosen which are related to the flux trial functions as follows:

$$\xi_k^j = F_0^T H_k, \quad \xi_i^{j*} = H_i^{*T} S^j, \quad K^j = K, \qquad \text{(A-22)}$$

where the subscript zero refers to an initial unperturbed or "reference" state. Then

$$I_{ik}^j = \int_R H_i^{*T} S^j F_0^T H_k dR, \qquad \text{(A-23)}$$

$$S_{ik}^j = \int H_i^{*T} S^j F_0^T H_k dR = I_{ik}^j, \qquad \text{(A-24)}$$

$$F_{ik}^{jT} = \int_R H_i^{*T} S^j F^T H_k dR. \qquad \text{(A-25)}$$

Furthermore, if the emergence spectrum of prompts and delays are the same, $S^j = X$, then

$$F_{ik}^{jT} = M_{ik} \qquad \text{(A-26)}$$

and

$$S_{ik}^j = M_{0ik}.$$

Making all these simplifications in Equations (A-10) and (A-11) and substituting Equation (A-11) into (A-10), one obtains:

$$\sum_{k=1}^{K} \left\{ [-(DB^2)_{ik} - A_{ik} + M_{ik}] T_k - \sum_{j=1}^{6} M_{0ik} \frac{d}{dt} \tau_k^j \right\} = \sum_{k=1}^{K} V_{ik}^{-1} \frac{d}{dt} T_k, \qquad \text{(A-27)}$$

$$\sum_{k=1}^{K} \left\{ \beta^j M_{ik} T_k - \lambda^j M_{0ik} \tau_k \right\} = \sum_{k=1}^{K} M_{0ik} \frac{d}{dt} \tau_k. \qquad \text{(A-28)}$$

One final simplification allows the precursor equations to be decoupled. If, in Equation (A-28), the difference between M_{ik} and M_{oik} is neglected (which amounts to ignoring the effect of perturbations in the fission cross section, insofar as they affect delayed neutron production), then the common matrix M_{oik} may be divided out of Equation (A-28), giving

$$\beta^i T_k - \lambda^i \tau_k = \frac{d}{dt} \tau_k \ . \qquad \text{(A-29)}$$

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